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UTILIZATION OF THE ACCESSORY MINIMUM PROBLEM IN TRAJECTORY ANALYSIS

By

Robert W. Hunt

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AEROBALLISTICS DIVISION

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SUMMARY

The accessory minimum problem is described as it occurs in the analysis of the "problem of Bolza" in the calculus of variations. A common trajectory optimization problem is stated in a form to which this theory may be directly applied. The specific equations arising in that application are written out, and a procedure for numerically testing trajectories (which at least satisfy Bliss's multiplier rule) for optimality is outlined.

INTRODUCTION

The general questions to be considered here are concerned with the investigation of the Jacobi necessary condition applied to trajectory optimization problems. This involves investigation of the accessory minimum problem in general and the definition, characterization, and identification of conjugate and focal points. This seems to be a hopeful area in which to work as the Jacobi equations form a well-behaved system of differential equations which can be investigated both analytically and numerically. The Jacobi necessary condition yields a further check on extremal arcs to those given by the other classical necessary conditions. It should also give further insight into the larger problem of sufficient conditions. The general outline of the investigation which follows is paralleled by a specific, illustrative problem.

General Problem (Bolza)

Find, in a class of piece-wise continuous arcs,

$$x_i(t) \qquad (i = 1, 2, \ldots, n;\ t_o \leq t \leq t_c) \qquad (1)$$

satisfying differential equations and end conditions,

$$\phi_\beta\ (t, x, \dot{x}) = 0 \qquad (\beta = 1, 2, \ldots, m < n) \qquad (2)$$

$$\psi_\mu\ \left[t_o, x(t_o), t_c, x(t_c)\right] = 0 \quad (\mu = 1, 2, \ldots\ p \leq 2n + 2), \qquad (3)$$

one which minimizes

$$J = g\left[t_o, x(t_o), t_c, x(t_c)\right] + \int_{t_o}^{t_c} f\ (t, x, \dot{x})\ dt. \qquad (4)$$

Here, the symbols x, $\dot{x}$ stand for the vectors

$$(x_1, x_2, \ldots, x_n), \quad (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n).$$

In parametric form,

$$J = \theta(\alpha) + \int_{t_o(\alpha)}^{t_c(\alpha)} f\ (t, x, \dot{x})\ dt, \qquad (5)$$

for the parameters $(\alpha) = (\alpha_1, \ldots, \alpha_r)$, $0 \leq \gamma \leq 2n + 2$, near (0) and admissible curves which join the end points $t_o\ (\alpha)$ and $t_c\ (\alpha)$ where the end conditions (3) are of the form

$$\begin{aligned} t_o &= t_o\ (\alpha_1, \ldots, \alpha_r), \quad x_i\ (t_o) = x_{io}\ (\alpha_1, \ldots, \alpha_r) \\ t_c &= t_c\ (\alpha_1, \ldots, \alpha_r) \quad\ x_i\ (t_c) = x_{ic}\ (\alpha_1, \ldots, \alpha_r) \end{aligned} \qquad (6)$$

for values of (α) near (o). Furthermore, the functions in (6) are of class C^2 and $\theta\ (\alpha)$ is any function of class C^2.

DEFINITION OF PROBLEM

Specific Problem

Find, in a class of piece-wise continuous arcs,

$$x_i\ (t) \qquad (i = 1, 2, \ldots, 5\ ;\ t_o \le t \le t_c) \tag{7}$$

satisfying the equations of motion,

$$\begin{aligned}
\phi_1 &\equiv \dot{x}_1 - \frac{F}{m}\cos\dot{x}_5 - \frac{K\,x_3}{(x_3^2 + x_4^2)^{\frac{3}{2}}} = 0 \\
\phi_2 &\equiv \dot{x}_2 - \frac{F}{m}\sin\dot{x}_5 - \frac{K\,x_4}{(x_3^2 + x_4^2)^{\frac{3}{2}}} = 0 \\
\phi_3 &\equiv \dot{x}_3 - x_1 = 0 \\
\dot{\phi}_4 &\equiv \dot{x}_4 - x_2 = 0,
\end{aligned} \tag{8}$$

and the end conditions,

$$\begin{aligned}
\psi_1 &= x_1(t_o) - x_{10} = 0 \\
\psi_2 &= x_2(t_o) - x_{20} = 0 \\
\psi_3 &= x_3(t_o) - x_{30} = 0 \\
\psi_4 &= x_4(t_o) - x_{40} = 0 \\
\psi_5 &= x_3^2(t_c) + x_4^2(t_c) - r_c^2 = 0 \\
\psi_6 &= x_1^2(t_c) + x_2^2(t_c) - \frac{r_o^2\,g}{r_c} = 0 \\
\psi_7 &= x_1(t_c)\ x_3(t_c) + x_2(t_c)\ x_4(t_c) = 0 \\
&\left(\psi_8 = \operatorname{Tan}^{-1}\frac{x_3(t_c)}{x_4(t_c)} - \phi_o - \omega t_c = o\right),
\end{aligned} \tag{9}$$

one which minimizes

$$J = \int_{t_o}^{t_c} dt \qquad (\text{or } J = t_c). \tag{10}$$

In this formulation, F, K, r_c, r_o, ϕ_o, and ω are given constants. The mass, m, is of the form $m_o + \dot{m}_o t$; and $\dot{m} = \dot{m}_o$ may be written as a constraining equation, but with no resulting gain. The control variable is $\dot{x}_5$, the direction of thrust.

In the parametric form, the only changes to the above formulation would be the parameterization of t_c, say

$$t_c = t_c + \alpha. \tag{11}$$

then

$$J = \int_{t_o}^{t_c + \alpha} dt \qquad (\text{or } J = \theta(\alpha) = t_c + \alpha). \tag{12}$$

This specific problem gives initial values for the position and velocity coordinates and specifies the position and velocity vectors at cut-off time, t_c. Furthermore, $\psi_7 = 0$ is the condition that the position and velocity vectors be orthogonal at cut-off time. The condition $\psi_8 = 0$, which may or may not be included, (as indicated by parentheses), specifies a position angle at t_c which is the constant ϕ_o if $\omega = o$ or $\phi_o + \omega t_c$ if $\omega \neq o$. This condition gives the problem a rendezvous formulation. Finally, the optimization considered here is for minimum cut-off time, i.e., minimum fuel consumption.

DISCUSSION

According to the multiplier rule, form

$$F\ (t, x, \dot{x}, \lambda) = 1 + \sum_{i=1}^{4} \lambda_i \phi_i, \tag{13}$$

and obtain the Euler-Lagrange equations. For the specific problem, these are

$$\begin{aligned}
\dot{\lambda}_3 &= \lambda_1 \left[\frac{-K}{r^3} + \frac{3K\, x_3^2}{r^5}\right] + \lambda_2 \frac{3K\, x_3\, x_4}{r^5} \\
\dot{\lambda}_4 &= \lambda_1 \frac{3K\, x_3\, x_4}{r^5} + \lambda_2 \left[\frac{-K}{r^3} + \frac{3K\, x_4^2}{r^5}\right] \\
\dot{\lambda}_1 &= -\lambda_3 \\
\dot{\lambda}_2 &= -\lambda_4 \\
\frac{F}{m} &(\lambda_1 \sin \dot{x}_5 - \lambda_2 \cos \dot{x}_5) = \text{constant}
\end{aligned} \tag{14}$$

along with the equations (8), where $r = (x_3^2 + x_4^2)^{1/2}$.

Every minimizing arc E must satisfy these equations, the equations (8), the end conditions (9), and the transversality condition. The latter can be written

$$\left[(F - \dot{x}_i F_{\dot{x}_i})\, dt + F_{\dot{x}_i}\, dx_i\right]_{t = t_c} + e_\mu\, d\Psi_\mu \equiv 0, \tag{15}$$

and must hold for some set of constants $\{e_\mu\}$ and for every choice of the differentials dt_c, dx_{ic}.

In the parametric formulation, the transversality condition is

$$\left[(F - \dot{x}_i F_{\dot{x}_i})\, dt + F_{\dot{x}_i}\, dx_i\right]_{t = t_c} + d\Theta \equiv 0. \tag{16}$$

The transversality condition for either form can be written out for the specific problem but will not be displayed here.

Thus, a system of first order differential equations and a set of boundary conditions are available, but with some of the boundary conditions being at the variable end point $t = t_c$. The necessary conditions of Weierstrass, Legendre, and Clebsch could now be investigated. These will not be discussed here, however, since this discussion is concerned only with the accessory problem and Jacobi's necessary condition.

Consider a one-parameter, admissible family of arcs

$$x_i\ (t,\ a) \qquad (t_o(a) \leq t \leq t_c(a),\ |a| < \epsilon). \tag{17}$$

The set of variations of the family along E is the set ξ_1, ξ_2, $\eta_i(t)$ defined by

$$\begin{aligned} dt_o &= t_{o_a}\ (o)\ da = \xi_1\ da \\ dt_c &= t_{c_a}\ (o)\ da = \xi_2\ da \\ dx_i &= \dot{x}_i\ dt + \delta x_i \\ \delta x_i &= x_{i_a}\ (t,\ o)\ da = \eta_i(t)\ da. \end{aligned} \tag{18}$$

In the parametric case previously considered, this could be accomplished by taking

$$\alpha_h = \alpha_h(a), \qquad \alpha_h(o) = o, \tag{19}$$

for a suitable set of functions.

If the arcs of an admissible family all satisfy the equations

$$\phi_\beta \ [t,\ x(t,\ a),\ \dot{x}(t,\ a)] = 0, \tag{20}$$

then the variations $\eta_i(t)$ along the arc E contained in the family for $a = 0$ satisfy

$$\Phi_\beta \ (t,\ \eta,\ \dot{\eta}) = \phi_{\beta_{x_i}} \eta_i + \phi_{\beta_{\dot{x}_i}} \dot{\eta}_i = 0. \tag{21}$$

For the specific problem, then,

$$\Phi_1 = \dot{\eta}_1 + \eta_3 \left[\frac{-K}{r^3} + \frac{3K\ x_3^2}{r^5}\right] + \eta_4 \left[\frac{3K\ x_3\ x_4}{r^5}\right] + \dot{\eta}\ \frac{F}{m} \sin \dot{x}_5 = 0$$

$$\Phi_2 = \dot{\eta}_2 + \eta_3 \left[\frac{3K\ x_3\ x_4}{r^5}\right] + \eta_4 \left[\frac{-K}{r^3} + \frac{3K\ x_4^2}{r^5}\right] - \dot{\eta}_5\ \frac{F}{m} \cos \dot{x}_5 = 0 \tag{22}$$

$$\Phi_3 = \dot{\eta}_3 - \eta_1 = 0$$

$$\Phi_4 = \dot{\eta}_4 - \eta_2 = 0$$

in which the x_i are the functions $x_i(t,\ o)$ belonging to E, the minimizing arc.

If the end values of the arcs of an admissible family satisfy the equations

$$\psi_\mu \left\{ t_o(a),\ x\left[t_o(a),\ a\right],\ t_c(a),\ x\left[t_c(a),\ a\right] \right\} = 0, \tag{23}$$

then the variations of the family along E satisfy

$$\Psi_\mu \left[\xi_1,\ \eta(t_o),\ \xi_2,\ \eta(t_c)\right] = \left(\psi_{\mu,o} + \dot{x}_{io}\ \psi_{\mu,\ io}\right) \xi_1$$

$$+ \psi_{\mu,io} \eta_i (t_o) + (\psi_{\mu,c} + \dot{x}_{ic}\ \psi_{\mu,ic})\ \xi_2 + \psi_{\mu,ic}\ \eta_i(t_c), \tag{24}$$

where $\psi_{\mu,o} = \frac{\partial \psi_\mu}{\partial x_o}$, $\psi_{\mu,io} = \frac{\partial \psi_\mu}{\partial x_{io}}$

and similarly for $\psi_{\mu,\ c}$ and $\psi_{\mu,\ ic}$.

For the specific problem, then,

$$
\begin{aligned}
\Psi_1 &= \dot{x}_{10}\,\xi_1 + \eta_1(t_o) = 0\\
\Psi_2 &= \dot{x}_{20}\,\xi_1 + \eta_2(t_o) = 0\\
\Psi_3 &= \dot{x}_{30}\,\xi_1 + \eta_3(t_o) = 0\\
\Psi_4 &= \dot{x}_{40}\,\xi_1 + \eta_4(t_o) = 0\\
\Psi_5 &= (2\,\dot{x}_{3c}\,x_{3c} + 2\,\dot{x}_{4c}\,x_{4c})\,\xi_2 + 2\,x_{3c}\,\eta_3(t_c) + 2\,x_{4c}\,\eta_4(t_c) = 0\\
\Psi_6 &= (2\,\dot{x}_{1c}\,x_{1c} + 2\,\dot{x}_{2c}\,x_{2c})\,\xi_2 + 2\,x_{1c}\,\eta_1(t_c) + 2\,x_{2c}\,\eta_2(t_c) = 0\\
\Psi_7 &= (\dot{x}_{1c}\,x_{3c} + \dot{x}_{2c}\,x_{4c} + \dot{x}_{3c}\,x_{1c} + x_{4c}\,x_{2c})\,\xi_2\\
&\quad + x_{3c}\,\eta_1(t_c) + x_{4c}\,\eta_2(t_c) + x_{1c}\,\eta_3(t_c) + x_{2c}\,\eta_4(t_c) = 0\\
\left(\psi_8\right. &= \left(\omega + x_{3c}\,x_{4c} - \frac{\dot{x}_{4c}\,x_{3c}}{x_{3c}^2 + x_{4c}^2}\right)\xi_2 + \frac{x_{4c}}{x_{3c}^2 + x_{4c}^2}\,\eta_3(t_c)\\
&\quad \left. - \frac{x_{3c}}{x_{3c}^2 + x_{4c}^2}\,\eta_4(t_c) = 0\right)
\end{aligned}
\tag{25}
$$

The first four of these can be simplified by taking $\xi_1 = 0$ since t_o is assumed to be fixed. These are the accessory end conditions and must be satisfied along with the accessory transversality conditions given as in (15), the integrand function now being a quadratic form $2\,\omega(\eta, \dot{\eta})$ to be defined presently.

Now consider the accessory minimum problem. The second variation of J is

$$
J_2(\xi, \eta) = 2\,\gamma\left[\xi_1, \eta_i(t_o), \xi_2, \eta_i(t_c)\right] + \int_{t_o}^{t_c} 2\,\omega(t, \eta, \dot{\eta})\,dt, \tag{26}
$$

where $2\,\gamma$ is a quadratic form in the variations at t_o and t_c and

$$
2\,\omega(t, \eta, \dot{\eta}) = F_{x_i x_k}\,\eta_i\,\eta_k + 2\,F_{x_i \dot{x}_k}\,\eta_i\,\dot{\eta}_k + F_{\dot{x}_i \dot{x}_k}\,\dot{\eta}_i\,\dot{\eta}_k, \tag{27}
$$

where the repeated sub-scripts are summed over.

For the parametric form, using (6) and (16) and defining the variations of x_i and α_h, respectively, to be

$$
\begin{aligned}
\eta_i(t) &= x_{i_a}(t, o)\\
u_h &= \alpha_h^1(o),
\end{aligned}
\tag{28}
$$

the second variation can be written

$$J_2(u, \eta) = b_{hk}\, u_h\, u_k + \int_{t_o}^{t_c} 2\,\omega(t, \eta, \dot{\eta})\, dt, \tag{29}$$

where

$$b_{hk} = \Bigg[(F - \dot{x}_i F_{\dot{x}_i}) \frac{\partial^2 t_c}{\partial \alpha_h \partial \alpha_k} + (F_t - \dot{x}_i F_{x_i}) \frac{\partial t_c}{\partial \alpha_h} \frac{\partial t_c}{\partial \alpha_k} + F_{x_i}\left(\frac{\partial t_c}{\partial \alpha_h} \frac{\partial x_{ic}}{\partial \alpha_k} + \frac{\partial t_c}{\partial \alpha_k} \frac{\partial x_{ic}}{\partial \alpha_h}\right) + F_{\dot{x}_i} \frac{\partial^2 x_{ic}}{\partial \alpha_h \partial \alpha_k}\Bigg]_{t = t_c} + \frac{\partial^2 \theta}{\partial \alpha_h \partial \alpha_k} \tag{30}$$

Furthermore, the secondary end conditions

$$\eta_i(t_c) = C_{i_h}\, u_h = \left[\frac{\partial x_{ic}}{\partial \alpha_h}(o) - \dot{x}_i(t_c) \frac{\partial t_c}{\partial \alpha_h}(o)\right] u_h \tag{31}$$

and the secondary transversality conditions

$$\omega_{\dot{\eta}_i}\, C_{i_h} \Big]_{t = t_c} + b_{hk}\, u_k = 0 \tag{32}$$

must be satisfied.

For the specific problem,

$$2\,\omega(t, \eta, \dot{\eta}) = \eta_3^2 \left[\frac{3K\,(3\lambda_1 x_3 + \lambda_2 x_4)}{r^5} - \frac{15\,K\,x_3^2\,(\lambda_1 x_3 + \lambda_2 x_4)}{r^7}\right] + 2\,\eta_3\,\eta_4 \left[\frac{3\,K\,(\lambda_1 x_4 + \lambda_2 x_3)}{r^5} - \frac{15\,K\,x_4^2\,(\lambda_1 x_3 + \lambda_2 x_4)}{r^7}\right] + \dot{\eta}_5^2 \left[\frac{F}{m}(\lambda_1 \cos \dot{x}_5 + \lambda_2 \sin \dot{x}_5)\right]. \tag{33}$$

For a minimizing arc, $J_2(\xi,\eta) \geq 0$. Thus, the accessory minimum problem is suggested. Find, in the class of admissible variations ξ_1, ξ_2, $\eta_i(t)$ satisfying the equations (22) and (25) along E, one which minimizes $J_2(\xi, \eta)$. This is equivalent in form to the initial problem. In some cases, the homogeneous quadratic form which is not under the integral sign in (26) or (29) is positive definite and leads to a consideration of minimizing $\int_{t_o}^{t_c} 2\,\omega(t, \eta, \dot{\eta})\,dt$ subject to all the proper conditions.

To proceed with the accessory minimum problem, let

$$\Omega\,(t, \eta, \dot{\eta}, \ell) = \omega\,(t, \eta, \dot{\eta}) + \sum_{i=1}^{4} \ell_i \;\Phi_i \,. \tag{34}$$

Then the Jacobi equations for the specific problem are

$$\begin{aligned}
\dot{\ell}_3 &= \eta_3 \left[\frac{3\,K\,(3\,\lambda_1\,x_3 + \lambda_2\,x_4)}{r^5} - \frac{15\,K\,x_3^2\,(\lambda_1\,x_3 + \lambda_2\,x_4)}{r^7}\right] \\
&+ \eta_4 \left[\frac{3\,K\,(\lambda_1\,x_4 + \lambda_2\,x_3)}{r^5} - \frac{15\,K\,x_3\,x_4\,(\lambda_1\,x_3 + \lambda_2\,x_4)}{r^7}\right] \\
&+ \ell_1 \left[\frac{-K}{r^3} + \frac{3\,K\,x_3^2}{r^5}\right] + \ell_2 \left[\frac{3\,K\,x_3\,x_4}{r^5}\right] \\
\dot{\ell}_4 &= \eta_3 \left[\frac{3\,K\,(\lambda_1\,x_4 + \lambda_2\,x_3)}{r^5} - \frac{15\,K\,x_4\,(\lambda_1\,x_3 + \lambda_2\,x_4)}{r^7}\right] \\
&+ \eta_4 \left[\frac{3\,K\,(3\,\lambda_2\,x_4 + \lambda_1\,x_3)}{r^5} - \frac{15\,K\,x_4^2\,(\lambda_1\,x_3 + \lambda_2\,x_4)}{r^7}\right] \\
&+ \ell_1 \left[\frac{3\,K\,x_3\,x_4}{r^5}\right] + \ell_2 \left[\frac{-K}{r^3} + \frac{3\,K\,x_4^2}{r^5}\right] \\
\dot{\ell}_1 &= -\,\ell_3 \\
\dot{\ell}_2 &= -\,\ell_4 \\
\dot{\eta}_5 \;& \frac{F}{m}\,(\lambda_1 \cos \dot{x}_5 + \lambda_2 \sin \dot{x}_5) + \frac{F}{m}\,(\ell_1 \sin \dot{x}_5 - \ell_2 \cos \dot{x}_5) = \text{constant,}
\end{aligned} \tag{35}$$

together with the equations (22). This problem has been formulated in such a way that the Jacobi system is non-singular, as can be seen by checking the determinant of the matrix $(F_{\dot{x}_i \dot{x}_j})$. These Jacobi equations, along with the equations (22), give a linear, homogeneous, first-order system of differential equations, a vector solution of which is of the form

$$\zeta(t) = \mathrm{col}\ (\eta_1\ \eta_2\ \eta_3\ \eta_4\ \eta_5\ \ell_1\ \ell_2\ \ell_3\ \ell_4).$$

By standard methods, a fundamental set of vector solutions can be defined by specifying appropriate initial conditions.

Now, a conjugate point (or focal point) t^* can be defined in the classical way as a value of t such that there exists a non-trivial vector solution $\zeta(t)$ of the Jacobi equations satisfying the transversality conditions and end conditions for $t = t_o$ and $t = t^*$. The word "conjugate" is usually used in the fixed end point case (when the transversality and end conditions are trivially satisfied) and the word "focal" when one or both end points vary over a manifold. Several characterizations of conjugate points or focal points in terms of certain determinants or matrices are then available.

The necessary condition of Jacobi states that if E affords a weak minimum to J, no conjugate point (or focal point) of $t = t_o$ on the open interval (t_o, t_c) can coincide with a point at which the Jacobi equations are non-singular.

For the specific problem, the interest now lies in analyzing a trajectory for focal points to $t = t_o$. This can be done as follows. First, a trajectory and cut-off time t_c is obtained numerically by using the Euler equations, the boundary conditions given, and arbitrary values for boundary conditions not specified or values previously ascertained to insure that the final cut-off conditions are satisfied. Then the accessory problem and its associated differential equations and conditions can be investigated numerically, taking advantage of the linearity of the Jacobi equations to obtain a fundamental family of vector solutions. Then, any of the several characterizations (1, 2) of conjugate and focal points can be used to check the Jacobi condition.

A number of experimental trajectories have been investigated with respect to the above concepts. In this manner, conjugate points have been found to exist, thus ruling out the associated trajectory as an optimum trajectory. In other cases, trajectories free of conjugate points have been obtained and thus retained as possible choices for the optimal trajectory corresponding to given conditions. The non-existence of conjugate points is still only a necessary, not a sufficient, condition for the minimizing arc. However, certain combinations of necessary conditions, Jacobi's among them, do yield sufficient conditions. This area is still open to much investigation and, in fact, provided some of the motivation for the present consideration of Jacobi's condition.

The procedure for searching for conjugate or focal points is still long and difficult. Hopefully, an iterative procedure can be worked out to check trajectories quickly and easily for such points. Future efforts in this research are to be directed toward this end as well as toward some information on sufficiency. Also, another closely allied area presently being investigated is concerned with the characteristic roots of the characteristic form of the accessory boundary problem. This will be discussed in a later report.

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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